A Polynomial Chaos Framework for Designing Linear Parameter Varying Control Systems

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Abstract

Here we use polynomial chaos framework to design controllers for linear parameter varying (LPV) dynamical systems. We assume the scheduling variable to be random and use polynomial chaos approach to synthesize the controller for the resulting linear stochastic dynamical system. The stability of the LPV system is formulated as an exponential mean-square (EMS) stability problem. Two algorithms are presented that guarantee EMS stability of the stochastic system and correspond to parameter dependent and independent Lyapunov functions, respectively. LPV controllers from the polynomial chaos based framework is shown to outperform LPV controller from classical design for an example nonlinear system.

I. Introduction

Linear parameter varying (LPV) systems are of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\rho)\boldsymbol{x} + \boldsymbol{B}(\rho)\boldsymbol{u}, \tag{1}$$

where system matrices depend on unknown parameter $\rho(t)$, which is measurable in real-time [1], [2]. Many nonlinear systems can be transformed to LPV systems and control systems can be designed using parameter dependent convex optimization problems. Typically, parameter dependent quantities are approximated using a known class of functions such as multilinear basis functions of ρ, linear fractional transformations of system matrices, or by gridding the parameter space. Both these approaches result in solution of a finite, but possible large, number of linear matrix inequalities (LMIs). Further, the choice of the basis functions or the resolution of the grid could lead to conservatisms in the design. Clearly, there is a tradeoff between problem size and conservatism in the design [3].

Fujisaki *et al.* [4] addressed the computational complexity of such problems by presenting a probabilistic approach to solve these problems, via a sequential randomized algorithm, which significantly reduces the computational complexity. Here the parameter $\rho(t)$ is assumed to be bounded i.e. $\rho(t) \in \mathcal{D}_\rho \subset \mathbb{R}^d$ and is treated as a random variable, with a distribution $f_\rho(\rho)$ defined over $\mathcal{D}_\rho$. The LPV synthesis problem is solved by sampling $\mathcal{D}_\rho$ and solving the sampled LMIs using a sequential-gradient method. As with any probabilistic algorithm, there is a tradeoff between sample complexity and confidence in the solution. Often, a large number of samples are required to generate a solution with high confidence. Also, the LMIs depend only on $\rho(t)$ and not in $\dot{\rho}(t)$ as it is in classical LPV formulation.

This paper is motivated by the work of Fujisaki *et al.* and is based on the idea of treating ρ as a random variable. Therefore, by substituting $\rho \equiv \boldsymbol{\Delta}$ in the system equation, we get

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{\Delta})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{\Delta})\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{\Delta} \in \mathbb{R}^d$ is a vector of uncertain parameters, with joint probability density function $f_{\boldsymbol{\Delta}}(\boldsymbol{\Delta})$. Matrices $\boldsymbol{A}(\boldsymbol{\Delta}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}(\boldsymbol{\Delta}) \in \mathbb{R}^{n\times m}$ are system matrices that depend on $\boldsymbol{\Delta}$. Consequently, the solution $\boldsymbol{x} := \boldsymbol{x}(t,\boldsymbol{\Delta}) \in \mathbb{R}^n$ also depends on $\boldsymbol{\Delta}$. Like in [4] we ignore temporal variation in the parameter and thus treat $\boldsymbol{\Delta}$ as random variables. Thus, we now study the system in (1) as a *linear time invariant system* with probabilistic system parameters. The LPV control design objective is then equivalent to designing a state-feedback law of the form $\boldsymbol{u} = \boldsymbol{K}(\boldsymbol{\Delta})\boldsymbol{x}$, which stabilizes the system in some suitable sense, where $\boldsymbol{K}(\boldsymbol{\Delta}) \in \mathbb{R}^{m\times n}$. Thus, we are looking to obtain a parameter dependent gain $\boldsymbol{K}(\boldsymbol{\Delta})$ that stabilizes the system in (2). The closed-loop system is then

$$\begin{aligned}\dot{\boldsymbol{x}} &= \left[\boldsymbol{A}(\boldsymbol{\Delta}) + \boldsymbol{B}(\boldsymbol{\Delta})\boldsymbol{K}(\boldsymbol{\Delta})\right]\boldsymbol{x},\\ &= \boldsymbol{A}_{cl}\boldsymbol{x}.\end{aligned} \tag{3}$$

There are two distinct differences between the work presented here and that in [4]. We do not use a randomized approach to solve the stochastic problem, and thus don't have issues related to confidence in the solution. In our approach, the stochastic problem is solved using polynomial chaos theory, which is a deterministic approach as described later. In addition, stability of the LPV system is formulated as an exponential mean square stability problem for the corresponding stochastic system. In [4], stability of the LPV system is formulated in the probabilistic sense. Computationally, the polynomial chaos framework is superior to sampling based approach in propagating uncertainty [5], and hence we can expect a computational advantage in using this framework to solve the stochastic formulation.

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Main contributions of this paper are two LPV controller synthesis algorithms with parameter dependent and independent Lyapunov functions respectively. They are presented as theorem 1 and 2. The paper is organized as follows. We first provide a brief background on polynomial chaos theory and show how it is applied to study linear dynamical systems with random parameters. This is followed by conditions for exponential mean-square stability in the polynomial chaos framework for closed-loop systems with parameter dependent controller. This leads to theorem 1 and 2. The paper ends with an example that highlights the superiority of the polynomial chaos approach over the classical LPV design approach.

II. Polynomial Chaos Theory

Polynomial chaos (PC) is a non-sampling based method to determine evolution of uncertainty in dynamical system, when there is probabilistic uncertainty in the system parameters. Polynomial chaos was first introduced by Wiener [6] where Hermite polynomials were used to model stochastic processes with Gaussian random variables. It can be thought of as an extension of Volterra's theory of nonlinear functionals for stochastic systems [7], [8]. According to Cameron and Martin [9] such an expansion converges in the $\mathcal{L}_2$ sense for any arbitrary stochastic process with finite second moment. This applies to most physical systems. Xiu *et al.* [10] generalized the result of Cameron-Martin to various continuous and discrete distributions using orthogonal polynomials from the so called Askey-scheme [11] and demonstrated $\mathcal{L}_2$ convergence in the corresponding Hilbert functional space. The PC framework has been applied to applications including stochastic fluid dynamics [12]–[14], stochastic finite elements [8], and solid mechanics [15], [16], feedback control [17]–[20] and estimation [21]. It has been shown that PC based methods are computationally far superior than Monte-Carlo based methods [5], [10], [12]–[14]. See [22] for several benchmark problems.

A general second order process $X(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ can be expressed by polynomial chaos as

$$X(\omega) = \sum_{i=0}^{\infty} x_i \phi_i(\mathbf{\Delta}(\omega)), \tag{4}$$

where ω is the random event and $\phi_i(\mathbf{\Delta}(\omega))$ denotes the polynomial chaos basis of degree p in terms of the random variables $\mathbf{\Delta}(\omega)$. $(\Omega, \mathcal{F}, P)$ is a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of the subsets of Ω, and P is the probability measure. According to Cameron and Martin [9] such an expansion converges in the $\mathcal{L}_2$ sense for any arbitrary stochastic process with finite second moment. In practice, the infinite series is truncated and $X(\omega)$ is approximated by

$$X(\omega) \approx \hat{X}(\omega) = \sum_{i=0}^{N} x_i \phi_i(\mathbf{\Delta}(\omega)).$$

The functions $\{\phi_i\}$ are a family of orthogonal basis in $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ satisfying the relation

$$\mathsf{E}\left[\phi_i \phi_j\right] := \int_{\mathcal{D}_{\mathbf{\Delta}}} \phi_i(\mathbf{\Delta}) \phi_j(\mathbf{\Delta}) f_{\mathbf{\Delta}}(\mathbf{\Delta})\, d\mathbf{\Delta} = h_i^2 \mathbf{\Delta}_{ij}, \tag{5}$$

where $\mathbf{\Delta}_{ij}$ is the Kronecker delta, h_i is a constant term corresponding to $\int_{\mathcal{D}_{\mathbf{\Delta}}} \phi_i^2 f_{\mathbf{\Delta}}(\mathbf{\Delta})\, d\mathbf{\Delta}$, $\mathcal{D}_{\mathbf{\Delta}}$ is the domain of the random variable $\mathbf{\Delta}(\omega)$, and $f_{\mathbf{\Delta}}(\mathbf{\Delta})$ is a probability density function for $\mathbf{\Delta}$. Table I shows the family of basis functions for random variables with common distributions.

Random Variable $\mathbf{\Delta}$	$\phi_i(\mathbf{\Delta})$ of the Wiener-Askey Scheme
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

TABLE I: Correspondence between choice of polynomials and given distribution of $\mathbf{\Delta}(\omega)$ [10].

A. Application to Dynamical Systems with Random Parameters

With respect to the dynamical system defined in (2), the solution can be approximated by the polynomial chaos expansion as

$$\boldsymbol{x}(t, \mathbf{\Delta}) \approx \hat{\boldsymbol{x}}(t, \mathbf{\Delta}) = \sum_{i=0}^{N} \boldsymbol{x}_i(t) \phi_i(\mathbf{\Delta}), \tag{6}$$

where the polynomial chaos coefficients $\boldsymbol{x}_i \in \mathbb{R}^n$. Define $\mathbf{\Phi}(\mathbf{\Delta})$ to be

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{\Delta}) := \begin{pmatrix} \phi_0(\mathbf{\Delta}) & \cdots & \phi_N(\mathbf{\Delta}) \end{pmatrix}^T, \text{ and} \tag{7}$$

$$\mathbf{\Phi}_n \equiv \mathbf{\Phi}_n(\mathbf{\Delta}) := \mathbf{\Phi}(\mathbf{\Delta}) \otimes \boldsymbol{I}_n, \tag{8}$$

where $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ is identity matrix. Also define matrix $\boldsymbol{X} \in \mathbb{R}^{n\times(N+1)}$, with polynomial chaos coefficients $\boldsymbol{x}_i$, as

$$\boldsymbol{X} = \begin{bmatrix}\boldsymbol{x}_0 & \cdots & \boldsymbol{x}_N\end{bmatrix}.$$

This lets us define $\hat{\boldsymbol{x}}(t, \boldsymbol{\Delta})$ as

$$\hat{\boldsymbol{x}}(t, \boldsymbol{\Delta}) := \boldsymbol{X}(t)\boldsymbol{\Phi}(\boldsymbol{\Delta}). \tag{9}$$

Noting that $\hat{\boldsymbol{x}} \equiv \textbf{vec}\,(\hat{\boldsymbol{x}})$, we obtain an alternate form for (9),

$$\hat{\boldsymbol{x}} \equiv \textbf{vec}\,(\hat{\boldsymbol{x}}) = \textbf{vec}\,(\boldsymbol{X}\boldsymbol{\Phi}) = \textbf{vec}\,(\boldsymbol{I}_n\boldsymbol{X}\boldsymbol{\Phi}) = (\boldsymbol{\Phi}^T \otimes \boldsymbol{I}_n)\textbf{vec}\,(\boldsymbol{X}) = \boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc}, \tag{10}$$

where $\boldsymbol{x}_{pc} := \textbf{vec}\,(\boldsymbol{X})$, and $\textbf{vec}\,(\cdot)$ is the vectorization operator [23].
Since $\hat{\boldsymbol{x}}$ from (10) is an approximation, substituting it in (3) we get equation error $\boldsymbol{e}$, which is given by

$$\boldsymbol{e} := \dot{\hat{\boldsymbol{x}}} - \boldsymbol{A}_{cl}(\boldsymbol{\Delta})\hat{\boldsymbol{x}} = \boldsymbol{\Phi}_n^T\dot{\boldsymbol{x}}_{pc} - \boldsymbol{A}_{cl}(\boldsymbol{\Delta})\boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc}. \tag{11}$$

Best $\mathcal{L}_2$ approximation is obtained by setting

$$\langle \boldsymbol{e}\phi_i\rangle := \mathsf{E}\left[\boldsymbol{e}\phi_i\right] = 0, \text{ for } i = 0, 1, \cdots, N. \tag{12}$$

$$\begin{aligned}
&\mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right]\dot{\boldsymbol{x}}_{pc} = \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}_{cl}\boldsymbol{\Phi}_n^T\right]\boldsymbol{x}_{pc},\\
&\implies \dot{\boldsymbol{x}}_{pc} = \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right]^{-1}\mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}_{cl}\boldsymbol{\Phi}_n^T\right]\boldsymbol{x}_{pc}, && (13)\\
&\text{or } \dot{\boldsymbol{x}}_{pc} = \boldsymbol{A}_{pc}\boldsymbol{x}_{pc}. && (14)
\end{aligned}$$

where $\boldsymbol{\Phi}_n$ and $\boldsymbol{A}_{cl}$ depend on $\boldsymbol{\Delta}$ as defined earlier.
We will need the following result in the rest of the paper.
Proposition 1: For any vector $\boldsymbol{v} \in \mathbb{R}^{N+1}$ and matrix $\boldsymbol{M} \in \mathbb{R}^{m\times n}$

$$\boldsymbol{M}(\boldsymbol{v}^T \otimes \boldsymbol{I}_n) = (\boldsymbol{v}^T \otimes \boldsymbol{I}_m)(\boldsymbol{I}_{N+1} \otimes \boldsymbol{M}), \tag{15}$$

where $\boldsymbol{I}_*$ is identity matrix with indicated dimension.

Proof:

$$\begin{aligned}
\boldsymbol{M}(\boldsymbol{v}^T \otimes \boldsymbol{I}_n) &= (1 \otimes \boldsymbol{M})(\boldsymbol{v}^T \otimes \boldsymbol{I}_n)\\
&= \boldsymbol{v}^T \otimes \boldsymbol{M} = (\boldsymbol{v}^T\boldsymbol{I}_{N+1}) \otimes (\boldsymbol{I}_m\boldsymbol{M})\\
&= (\boldsymbol{v}^T \otimes \boldsymbol{I}_m)(\boldsymbol{I}_{N+1} \otimes \boldsymbol{M}).
\end{aligned}$$

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III. Controller Synthesis

The controller gain $\boldsymbol{K}(\boldsymbol{\Delta})$ can be introduced in the polynomial chaos framework by substituting $\boldsymbol{A}_{cl} := \boldsymbol{A}(\boldsymbol{\Delta}) + \boldsymbol{B}(\boldsymbol{\Delta})\boldsymbol{K}(\boldsymbol{\Delta})$, in (13) to get

$$\dot{\boldsymbol{x}}_{pc} = \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right]^{-1}\left(\mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}\boldsymbol{\Phi}_n^T\right] + \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{B}\boldsymbol{K}\boldsymbol{\Phi}_n^T\right]\right)\boldsymbol{x}_{pc} \tag{16}$$

Polynomial chaos expansion of $\boldsymbol{K}(\boldsymbol{\Delta})$ can be written as

$$\begin{aligned}
\boldsymbol{K}(\boldsymbol{\Delta}) &= \sum_{i=0}^{N}\boldsymbol{K}_i\phi_i(\boldsymbol{\Delta}), \boldsymbol{K}_i \in \mathbb{R}^{m\times n};\\
&= \begin{bmatrix}\phi_0\boldsymbol{I}_m & \cdots & \phi_N\boldsymbol{I}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{K}_0\\ \vdots\\ \boldsymbol{K}_N\end{bmatrix},\\
&= (\boldsymbol{\Phi}^T \otimes \boldsymbol{I}_m)\boldsymbol{V}_{\boldsymbol{K}} = \boldsymbol{\Phi}_m^T\boldsymbol{V}_{\boldsymbol{K}}, && (17)
\end{aligned}$$

where $\boldsymbol{V}_{\boldsymbol{K}} \in \mathbb{R}^{m(N+1)\times n}$ is the vertical stacking of $\boldsymbol{K}_i$. The expression $\boldsymbol{B}\boldsymbol{K}\boldsymbol{\Phi}_n^T$ in (16) can be simplified using (15) as

$$\boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{V}_{\boldsymbol{K}}\boldsymbol{\Phi}_n^T = \boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\mathcal{V}_{\boldsymbol{K}},$$

where $\mathcal{V}_{\boldsymbol{K}} := \boldsymbol{I}_{N+1} \otimes \boldsymbol{V}_{\boldsymbol{K}}$. Therefore,

$$\dot{\boldsymbol{x}}_{pc} = \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right]^{-1}\left(\mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}\boldsymbol{\Phi}_n^T\right] + \mathsf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\right]\mathcal{V}_{\boldsymbol{K}}\right)\boldsymbol{x}_{pc}. \tag{18}$$

Recall that for the dynamical system in (3), the equilibrium solution is said to possess exponential stability of the m^{th} mean if $\exists\, \boldsymbol{\Delta} > 0$ and constants $\alpha > 0, \beta > 0$ such that $\|\boldsymbol{x}_0\| < \boldsymbol{\Delta}$ implies $\forall t \geq t_0$ [24], [25]

$$\mathbf{E}\left[\|\boldsymbol{x}(t;\boldsymbol{x}_0,t_0)\|_m^m\right] \leq \beta \mathbf{E}\left[\|\boldsymbol{x}_0\|_m^m\right] e^{-\alpha(t-t_0)}. \tag{19}$$

It can be shown [26] that the dynamical system in (3), with random variables $\boldsymbol{\Delta}$, is exponentially stable in the 2^{nd} mean, or exponentially stable in the mean square sense (EMS-stable), if $\exists$ a Lyapunov function $V(\boldsymbol{x}) := \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$, with $\boldsymbol{P} = \boldsymbol{P}^T > 0$, and $\alpha > 0$ such that

$$\mathbf{E}\left[\dot{V}\right] \leq -\alpha \mathbf{E}\left[V\right]. \tag{20}$$

Theorem 1: The closed-loop system (3) is EMS-stable with controller $\boldsymbol{K}(\boldsymbol{\Delta})$ if $\exists\ \boldsymbol{P} = \boldsymbol{P}^T > 0$ and $\alpha > 0$ such that

$$\mathcal{Y}\mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}\boldsymbol{\Phi}_n^T\right]^T + \mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}\boldsymbol{\Phi}_n^T\right]\mathcal{Y} + \mathcal{W}^T\mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m\times(N+1)}^T\right]^T + \mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m\times(N+1)}^T\right]\mathcal{W} + \alpha\mathcal{Y}\mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right] \leq 0, \tag{21}$$

where $\mathcal{W} := \boldsymbol{I}_{N+1} \otimes \boldsymbol{W}$, $\mathcal{Y} := \boldsymbol{I}_{N+1} \otimes \boldsymbol{Y}$, $\boldsymbol{Y} := \boldsymbol{P}^{-1}$, and $\boldsymbol{W} := \boldsymbol{V}_{\boldsymbol{K}}\boldsymbol{Y}$.

Proof: With $V(\boldsymbol{x}) := \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$, and $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\dot{V} = \dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\dot{\boldsymbol{x}}$. The term $\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x}$ can be approximated by the polynomial chaos expansion as

$$\begin{aligned}
&\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x}\\
&\approx \boldsymbol{x}_{pc}^T\left[\left(\boldsymbol{A}\boldsymbol{\Phi}_n^T + \boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\mathcal{V}_{\boldsymbol{K}}\right)^T\boldsymbol{P}\boldsymbol{\Phi}_n^T\right]\boldsymbol{x}_{pc},\\
&= \boldsymbol{x}_{pc}^T\left(\boldsymbol{\Phi}_n\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{\Phi}_n^T + \mathcal{V}_{\boldsymbol{K}}^T\boldsymbol{\Phi}_{m(N+1)}\boldsymbol{\Phi}_m\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{\Phi}_n^T\right)\boldsymbol{x}_{pc},
\end{aligned}$$

Using (15) we can write $\boldsymbol{P}\boldsymbol{\Phi}_n^T = \boldsymbol{\Phi}_n^T\mathcal{P}$, where $\mathcal{P} := \boldsymbol{I}_{N+1} \otimes \boldsymbol{P}$. Substituting them in $\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x}$ we get

$$\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x} \approx \boldsymbol{x}_{pc}^T\left(\boldsymbol{\Phi}_n\boldsymbol{A}^T\boldsymbol{\Phi}_n^T\mathcal{P} + \mathcal{V}_{\boldsymbol{K}}^T\boldsymbol{\Phi}_{m(N+1)}\boldsymbol{\Phi}_m\boldsymbol{B}^T\boldsymbol{\Phi}_n^T\mathcal{P}\right)\boldsymbol{x}_{pc}. \tag{22}$$

The Lyapunov function $V := \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ can be written as

$$V := \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}_{pc}^T\boldsymbol{\Phi}_n\boldsymbol{P}\boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc} = \boldsymbol{x}_{pc}^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P}\boldsymbol{x}_{pc}.$$

Therefore, $\mathbf{E}\left[\dot{V}\right] \leq -\alpha\mathbf{E}\left[V\right]$ is equivalent to

$$\mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{A}^T\boldsymbol{\Phi}_n^T\right]\mathcal{P} + \mathcal{V}_{\boldsymbol{K}}^T\mathbf{E}\left[\boldsymbol{\Phi}_{m(N+1)}\boldsymbol{\Phi}_m\boldsymbol{B}^T\boldsymbol{\Phi}_n^T\right]\mathcal{P} + (*)^T \leq -\alpha\mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\right]\mathcal{P},$$

where $(*)^T$ are the symmetric terms. The above BMI can be convexified using the well known substitutions [27] $\boldsymbol{Y} := \boldsymbol{P}^{-1}$, and $\boldsymbol{W} := \boldsymbol{V}_{\boldsymbol{K}}\boldsymbol{Y}$. These substitutions can be written in terms of $\mathcal{P}, \mathcal{V}_{\boldsymbol{K}}, \mathcal{Y}$, and $\mathcal{W}$ as

$$\begin{aligned}
\mathcal{W} &= \boldsymbol{I}_{N+1} \otimes \boldsymbol{W} = \boldsymbol{I}_{N+1}\boldsymbol{I}_{N+1} \otimes \boldsymbol{V}_{\boldsymbol{K}}\boldsymbol{Y}\\
&= (\boldsymbol{I}_{N+1} \otimes \boldsymbol{V}_{\boldsymbol{K}})(\boldsymbol{I}_{N+1} \otimes \boldsymbol{Y})\\
&= \mathcal{V}_{\boldsymbol{K}}\mathcal{Y}.
\end{aligned}$$

It is also straightforward to show $\mathcal{P} = \mathcal{Y}^{-1}$ and $\mathcal{V}_{\boldsymbol{K}} = \mathcal{W}\mathcal{Y}^{-1}$. Substituting these in the above BMI, and pre-post multiplying by $\mathcal{Y}$, we get the result. ■

Theorem 2: The closed-loop system (3) is EMS-stable with controller $\boldsymbol{K}(\boldsymbol{\Delta})$ if $\exists\ \boldsymbol{P}(\boldsymbol{\Delta}) = \boldsymbol{P}^T(\boldsymbol{\Delta}) > 0$ and $\alpha > 0$ such that

$$\mathcal{Y}\boldsymbol{M}_1^T + \boldsymbol{M}_1\mathcal{Y} + \mathcal{W}^T\boldsymbol{M}_2^T + \boldsymbol{M}_2\mathcal{W} + \alpha\mathcal{Y}\boldsymbol{M}_0 \leq 0, \tag{23}$$

where

$$\begin{aligned}
&\boldsymbol{P}(\boldsymbol{\Delta}) := \boldsymbol{\Phi}_n^T(\boldsymbol{\Delta})\mathcal{P}\boldsymbol{\Phi}_n(\boldsymbol{\Delta}),\\
&\mathcal{P} := \boldsymbol{I}_{N+1} \otimes \boldsymbol{P}_0, \boldsymbol{P}_0 = \boldsymbol{P}_0^T > 0,\\
&\boldsymbol{M}_0 = \mathbf{E}\left[(\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T)^2\right],\\
&\boldsymbol{M}_1 = \mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\boldsymbol{\Phi}_n\boldsymbol{A}\boldsymbol{\Phi}_n^T\right],\\
&\boldsymbol{M}_2 = \mathbf{E}\left[\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\boldsymbol{\Phi}_n\boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\right].
\end{aligned}$$

Proof: Define Lyapunov function $V(\boldsymbol{x}) := \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{\Delta})\boldsymbol{x}$, with $\boldsymbol{P}(\boldsymbol{\Delta}) := \boldsymbol{\Phi}_n^T(\boldsymbol{\Delta})\mathcal{P}\boldsymbol{\Phi}_n(\boldsymbol{\Delta})$, $\mathcal{P} := \boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0$, and $\boldsymbol{P}_0 = \boldsymbol{P}_0^T > 0$. The form for $\boldsymbol{P}(\boldsymbol{\Delta})$ is motivated by the literature on sum-of-square representation of matrix polynomials [28], [29], which ensures $\boldsymbol{P}(\boldsymbol{\Delta}) > 0$. The Lyapunov function can be simplified as

$$\begin{aligned}
V(\boldsymbol{x}) &= \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{\Delta})\boldsymbol{x}\\
&\approx \boldsymbol{x}_{pc}^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T(\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T(\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}^T((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{I}_n)(\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{I}_n)\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}^T((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{I}_n)((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{P}_0)\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}^T((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)^2\otimes\boldsymbol{I}_n)(\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}^T(\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T)^2\mathcal{P}\boldsymbol{x}_{pc}.
\end{aligned}$$

$\dot{V} = \dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\dot{\boldsymbol{x}}$. The term $\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x}$ can be approximated by the polynomial chaos expansion as

$$\dot{\boldsymbol{x}}^T\boldsymbol{P}(\boldsymbol{\Delta})\boldsymbol{x} \approx \boldsymbol{x}_{pc}^T\left(\boldsymbol{A}\boldsymbol{\Phi}_n^T + \boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\mathcal{V}_{\boldsymbol{K}}\right)^T\boldsymbol{\Phi}_n^T\mathcal{P}\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\boldsymbol{x}_{pc}.$$

We next show that $\mathcal{P}\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T = \boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P}$.

$$\begin{aligned}
\mathcal{P}\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T &= (\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)(\boldsymbol{\Phi}\otimes\boldsymbol{I}_n)(\boldsymbol{\Phi}^T\otimes\boldsymbol{I}_n)\\
&= (\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{I}_n)\\
&= (\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{P}_0\\
&= (\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\boldsymbol{I}_{N+1}\otimes\boldsymbol{I}_n\boldsymbol{P}_0\\
&= ((\boldsymbol{\Phi}\boldsymbol{\Phi}^T)\otimes\boldsymbol{I}_n\boldsymbol{I}_n)(\boldsymbol{I}_{N+1}\otimes\boldsymbol{P}_0)\\
&= \boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\dot{\boldsymbol{x}}^T\boldsymbol{P}(\boldsymbol{\Delta})\boldsymbol{x}\\
&\approx \boldsymbol{x}_{pc}^T\left(\boldsymbol{A}\boldsymbol{\Phi}_n^T + \boldsymbol{B}\boldsymbol{\Phi}_m^T\boldsymbol{\Phi}_{m(N+1)}^T\mathcal{V}_{\boldsymbol{K}}\right)^T\boldsymbol{\Phi}_n^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P}\boldsymbol{x}_{pc}\\
&= \boldsymbol{x}_{pc}\left(\boldsymbol{\Phi}_n\boldsymbol{A}^T\boldsymbol{\Phi}_n^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P} + \mathcal{V}_{\boldsymbol{K}}^T\boldsymbol{\Phi}_{m(N+1)}\boldsymbol{\Phi}_m\boldsymbol{B}^T\boldsymbol{\Phi}_n^T\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^T\mathcal{P}\right)\boldsymbol{x}_{pc}.
\end{aligned}$$

Therefore, $\mathbf{E}\left[\dot{V}\right] \leq -\alpha\mathbf{E}\left[V\right]$ is equivalent to

$$\boldsymbol{M}_1^T\mathcal{P} + \mathcal{P}\boldsymbol{M}_1 + \mathcal{V}_{\boldsymbol{K}}^T\boldsymbol{M}_2^T\mathcal{P} + \mathcal{P}\boldsymbol{M}_2\mathcal{V}_{\boldsymbol{K}} + \alpha\boldsymbol{M}_0\mathcal{P} \leq 0,$$

which can be convexified as in Theorem 1 to obtain the result. ■

IV. Example

Here we consider the control of the following nonlinear system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}0 & 1\\ -1 & (1-x_1^2)\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u. \tag{24}$$

The above systems is the Van der Pol oscillator with a control input. We transform it to an LPV system by introducing the parameter $\rho := 1 - x_1^2$. The objective is to design a state feedback controller $K(\rho)$ that will quadratically stabilize the above system. We restrict stabilization of the set defined by $\boldsymbol{x} \in [-5,5]^2$. Therefore, $\rho \in [-24,1]$. For the PC LPV algorithm, we assume $\rho \equiv \boldsymbol{\Delta} \in \mathcal{U}_{[-24,1]}$, a uniformly distributed random variable over $[-24,1]$. Fig.(1) shows the state and control trajectories of (24) with three control systems $\boldsymbol{K}_{\text{LTI}}, \boldsymbol{K}_{\text{LPV}}(\rho)$ and $\boldsymbol{K}_{\text{pcLPV}}(\boldsymbol{\Delta})$, and were designed with $\alpha = 1$ in the following manner:

- $\boldsymbol{K}_{\text{LTI}}$, from linearized dynamics ($A_{\text{LTI}}, B_{\text{LTI}}$) about $(0,0)$, satisfying

$$\boldsymbol{Y}_{\text{LTI}}\boldsymbol{A}_{\text{LTI}}^T + \boldsymbol{A}_{\text{LTI}}\boldsymbol{Y}_{\text{LTI}} + \boldsymbol{W}_{\text{LTI}}^T\boldsymbol{B}_{\text{LTI}}^T + \boldsymbol{B}_{\text{LTI}}\boldsymbol{W}_{\text{LTI}} + \alpha\boldsymbol{Y}_{\text{LTI}} \leq 0.$$

- $\boldsymbol{K}_{\text{LPV}}(\rho)$, from LPV dynamics

$$\boldsymbol{A}_{\text{LPV}}(\rho) := \begin{bmatrix}0 & 1\\ -1 & \rho\end{bmatrix}, \boldsymbol{B}_{\text{LPV}}(\rho) := \begin{bmatrix}0\\ 1\end{bmatrix}$$

satisfying

$$\boldsymbol{Y}_{\text{LPV}}(\rho_k)\boldsymbol{A}_{\text{LPV}}^T(\rho_k)+\boldsymbol{A}_{\text{LPV}}(\rho_k)\boldsymbol{Y}_{\text{LPV}}(\rho_k)+\boldsymbol{W}_{\text{LPV}}^T(\rho_k)\boldsymbol{B}_{\text{LPV}}^T(\rho_k)+\boldsymbol{B}_{\text{LPV}}(\rho_k)\boldsymbol{W}_{\text{LPV}}(\rho_k)+\alpha\boldsymbol{Y}_{\text{LPV}}(\rho_k)\leq 0,$$

where

$$\boldsymbol{Y}_{\text{LPV}}(\rho_k):=\boldsymbol{Y}_0+\rho_k\boldsymbol{Y}_1>0,\boldsymbol{Y}_i=\boldsymbol{Y}_i^T,$$

and ρ_k are the samples from $\mathcal{U}_{[-24,1]}$.

- $\boldsymbol{K}_{\text{pcLPV}}$, from theorem 2, assuming $\rho\equiv\boldsymbol{\Delta}\in\mathcal{U}_{[-24,1]}$.

Fig.(1) shows the state and control trajectories of the nonlinear closed-loop system for initial condition $(5,5)$. $\boldsymbol{K}_{\text{pcLPV}}$ is designed with *first order* polynomial chaos expansion and several $\boldsymbol{K}_{\text{LPV}}$s are designed with $2,5,10$ and 50 samples from $\mathcal{U}_{[-24,1]}$.

For this problem, we make the following observations.

1) Increasing the order of the polynomial chaos expansion does not significantly improve controller performance. We are able to achieve high performance with very low order polynomial chaos expansion.
2) As seen from fig.(1) increasing the number of samples in the design of $\boldsymbol{K}_{\text{LPV}}$, improves the performance, but doesn't quite reach the performance of $\boldsymbol{K}_{\text{pcLPV}}$. The computational times for $\boldsymbol{K}_{\text{LPV}}$ synthesis are as follows:

Controller	Synthesis Time (s)
$\boldsymbol{K}_{\text{pcLPV}}$ (first order PC)	0.3556
$\boldsymbol{K}_{\text{LPV}}$ (2 samples)	0.4008
$\boldsymbol{K}_{\text{LPV}}$ (5 samples)	0.5463
$\boldsymbol{K}_{\text{LPV}}$ (10 samples)	0.6723
$\boldsymbol{K}_{\text{LPV}}$ (50 samples)	1.9943

   Thus $\boldsymbol{K}_{\text{pcLPV}}$ has a clear advantage over sampled based $\boldsymbol{K}_{\text{LPV}}$ design in terms of controller performance and computational complexity.
3) Both $\boldsymbol{K}_{\text{pcLPV}}$ and $\boldsymbol{K}_{\text{LPV}}$ outperform $\boldsymbol{K}_{\text{LTI}}$ as expected.

The controllers were synthesized in `MATLAB` [30] using `CVX` [31].

V. Summary

In this paper we presented a new framework to synthesize LPV controllers using polynomial chaos framework. This framework builds on the probabilistic representation of the scheduling variable and the synthesis was done by treating the LPV system as a stochastic linear system. Two synthesis algorithms were presented which correspond to parameter dependent and independent Lyapunov functions. The algorithms were tested on a nonlinear dynamical system and outperformed controllers synthesized using classical LPV design techniques.

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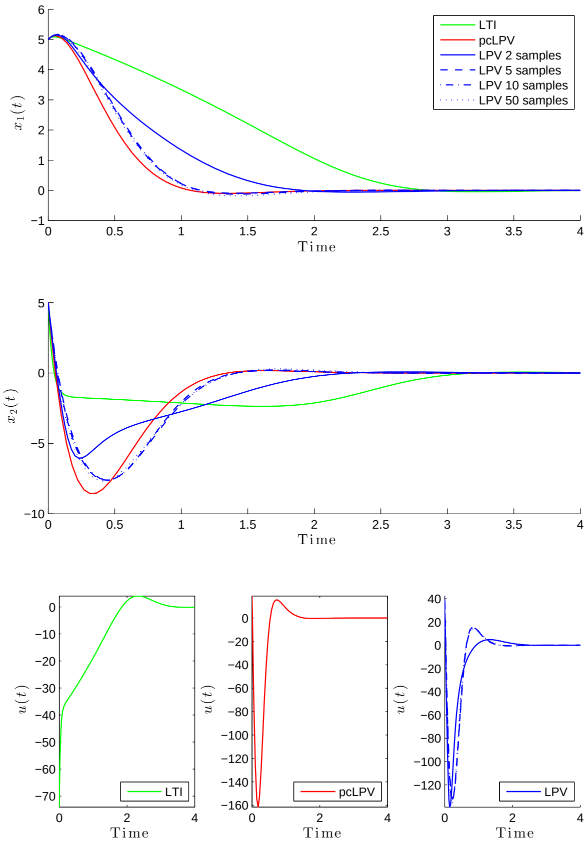

Fig. 1: State and control trajectories

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